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Fiber-complemented graphs II. Retractions and endomorphisms

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Abstract

A *fiber-complemented* graph is a graph for which the inverse image of every prefiber (or gated set) by any projection map onto a prefiber is a prefiber. In this paper, we continue the study of these graphs and establish a retraction theorem and fixed point properties for endomorphisms. Adding the notion of *mooring* (these are particular retractions of a graph onto its prefibers) to the tools introduced in Part I of this work (Discrete Math. 226 (2001) 107), we show that a fiber-complemented graph whose elementary prefibers induce moorable graphs is a retract of a Cartesian product of elementary moorable graphs. Then we deduce that under some conditions of compacticity, the elements of every commuting family of endomorphisms of a moorable pre-median graph strictly stabilize a nonempty finite pre-median subgraph (*pre-median graphs* are particular instances of weakly modular graphs which are fiber-complemented). These results give generalizations of analogous properties related to median graphs, quasi-median graphs, pseudo-median graphs and weakly median graphs.

Keywords: Median graph; Quasi-median graph; Prefiber; Retraction; Prefiber; Fixed-point property

1. Introduction

A subset A of a metric space (X,d) is a *prefiber* (or a *gated set*) of X if, for every $x \in X$, there exists $y \in A$ such that $d(x,z) = d(x,y) + d(y,z)$ for every $z \in A$. The element y, necessarily unique for every x, defines a projection map proj_A of X onto the

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prefiber A. In [10], we defined the *fiber-complemented* graphs as graphs which have the *fiber-complementation property*:

> The inverse image of every prefiber by any projection map onto a prefiber is a prefiber.

From this property we deduced: (1) a procedure of construction by amalgamation or by expansion where the minimal prefibers with respect to inclusion (called *elementary prefibers*) work like building stones and depend on each class of graphs; (2) a theorem of canonical isometric embedding into a Cartesian product of graphs whose factors, called *elementary graphs*, are induced by elementary prefibers; (3) a characterization of compact fiber-complemented graphs with respect to the topology induced by the prefibers as closed sets, which implies that every fiber-complemented graph without isometric ray contains a Cartesian product of elementary graphs which is invariant under every automorphism. Furthermore, we introduced the class of pre-median graphs. These are weakly modular graphs not containing the graph $K_{2,3}$ and the graph $K_{2,3}$ with an extra edge as induced subgraphs. We proved that every pre-median graph is fiber-complemented. As consequences, this theory gives a global approach to obtain several previous results related to median graphs, quasi-median graphs, pseudo-median graphs, weakly median graphs, bridged graphs, and extend them to the infinite case, since these graphs are particular instances of pre-median graphs.

We continue this description of fiber-complemented graphs started in [10] with the study of fixed point properties for endomorphisms, which are, for such a graph G, maps of $V(G)$ into $V(G)$ which preserve or contract the edges of G. In this paper, our purpose is to obtain generalizations of two properties of median graphs. First, Bandelt established in [1] that the median graphs are the retracts of hypercubes. A homomorphism $f : H \to G$ (or a contraction; see [10, Section 2.3]) is a *retraction* whenever there exists a homomorphism $g : G \to H$ such that $f \circ g$ is the identity map on G; under this condition, G is said to be a *retract* of H and usually it is convenient to identify G as an isometric subgraph of H. The study of classes of graphs closed under retractions and products (the *graph varieties*) was initiated mainly by Hell [19] and Rival [22]; however they considered a product of graphs, called *strong product*, different from the Cartesian product used in this paper. Besides we mention that these authors dealt with a slightly different notion of retractions, where the morphisms are the edge-preserving maps, and one could say that our own retracts are *reflexive* (a stricter concept would be useless because the presence of triangles in fiber-complemented graphs). The above-mentioned result of Bandelt was generalized to quasi-median graphs by Wilkeit [27] for the finite case and by Chastand [9] for the infinite case; in this paper we extend it to a large class of fiber-complemented graphs which contains in particular weakly median graphs.

In order to get these generalizations, we use several results obtained in Part I of our work [10] and to these basic ingredients, we need to add the concept of *mooring* of a fiber-complemented graph G with respect to some given prefiber W. A endomorphism φ of G is a *mooring* of G onto W if $\varphi(x) = x$ for every x in W and $\{x, \varphi(x)\}$ is an edge of $I(x, \mathrm{proj}_W(x))$ otherwise, and a graph G is *moorable* if there exists a mooring onto every vertex of G. The existence of such morphisms enables us to give sufficient conditions to establish our first main result:

Theorem 1. *Every moorable fiber-complemented graph G is a retract of the Cartesian product of the graphs induced by representatives of its classes of elementary prefibers with respect to the parallelism relation.*

Another result, due to Bandelt and van de Vel [7], states that for each endomorphism of a finite median graph G there is a cube $Q \subseteq G$ with $f(Q) = Q$ (in this case, one says that Q is *strictly stabilized* under f). The topic of fixed point properties in graphs was initiated by Tits [26] for trees and Halin [18] for connected locally finite graphs. Several similar results were obtained afterwards, with respect to various classes of graphs in which one considers two kinds of families of contractions, the group of automorphisms of a graph, or any commuting family of endomorphisms (see Polat [20] for a survey). In this field the above-mentioned result of Bandelt and van de Vel on median graphs was improved by Tardif [24], who established that every commuting family of endomorphisms of any infinite median graph without isometric rays stabilizes a common finite hypercube and recently analogous properties were given by Polat [21] for some other classes of graphs. Moreover in [10] we proved an invariant subgraph property in compact fiber-complemented graphs by means of the characterization of such compact graphs with respect to the topology generated by the prefibers. Thus we complete this work with theorems which generalize several results on median graphs, quasi-median graphs, pseudo-median graphs, weakly median graphs in [7,5,12,24,2] (a *box* is an induced subgraph which is isomorphic to a Cartesian product of connected graphs).

Theorem 2. *A moorable pre-median graph G without infinite elementary prefibers has the fixed box property* (*i.e.*, *every endomorphism of G strictly stabilizes a finite nonempty pre-median box*) *if and only if it contains no isometric rays.*

Theorem 3. *If G is a moorable pre-median graph without isometric rays and infinite elementary prefibers*, *then the elements of every commuting family of endomorphisms strictly stabilize a common finite nonempty pre-median box.*

2. Preliminaries

Here we omit to summarize the general definitions and notations introduced in Part I and we incite the reader to refer to [10].

2.1. Moorable graphs

2.1.1. Definition. Let W be a prefiber of a graph G. An endomorphism φ of G is a *p-mooring* (or simply a *mooring*) of G onto W if $\varphi(x) = x$ for every x in W and $\{x, \varphi(x)\}$ is an edge of $G[I(x, \mathrm{proj}_W(x))]$ otherwise.

A graph G is *moorable* if, for every vertex u of G, there exists a mooring of G onto $\{u\}$ (or simply onto u).

2.1.2. Remark.

1. If φ is a mooring of G onto the prefiber W and if $d_G(x,W)=n$ for $x\in V(G)$, then $\varphi^n(x)=\mathrm{proj}_W(x)$.
2. This definition is slightly different of that of Tardif in [24], who only imposed that $\{x,\varphi(x)\}\in E(G)$ for every $x\in V(G-W)$. Nevertheless both definitions agree on median graphs since median graphs are triangle-free.
3. Obviously, the existence of moorings onto every prefiber of some graph implies that it is moorable, but the converse remains an open problem. However, the equivalence holds naturally in any elementary graph and below (Theorem 3.3.1) we will also prove it in the case of fiber-complemented graphs.
4. In a breadth-first search (or BFS) of a given graph G from a vertex u, the *father* $\varphi(x)$ of any $x\in V(G)$ is the first neighbour of x which is visited. Clearly, for every vertex x of G, $d_G(u,x)=d_G(u,\varphi(x))+1$. Thus, φ is a mooring of G onto u whenever it is an endomorphism of $V(G)$.
   Consequently, if for every $u\in V(G)$, there exists a BFS whose father function is an endomorphism, then G is a moorable graph. In [11], Chastand, Laviolette and Polat established that in a bridged graph, the father function of any BFS is an endomorphism.

2.1.3. Proposition (Chastand et al. [11, Proposition 4.4]). *Every bridged graph is moorable.*

We also mention that this result is implicit in [14,15]. For an endomorphism φ of a graph G and a vertex u of G, it is easy to check that the family of subgraphs induced by the sets $\{\varphi^n(x)\colon\ n\geqslant 0\}$ for every $x\in V(G)$ is a *geodesic* 1-*combing* with basepoint u, as defined by Chepoi in [15], if and only if φ is a mooring of G onto u. Thus, Propositions 9.1 and 9.4 in [15] imply that every bridged graph and every Helly graph are moorable.

2.2. *Pre-median graphs*

In [8], we introduced the class of pre-median graphs which are weakly modular graphs without two particular graphs as induced subgraphs: the complete bipartite graph $K_{2,3}$ (Fig. 1) and the complete bipartite graph $K_{2,3}$ with an extra edge (Fig. 2).

We recall that we proved in [8,10] that every pre-median graph is fiber-complemented and the class of pre-median graphs contains the class of weakly median graphs (and its subclasses: pseudo-median graphs, quasi-median graphs, median graphs; see [3,5,6,2]) and the class of bridged graphs (see [16,17] for some topics on these graphs and [10] for the main features which are suitable for the fiber-complementation property). Thus, all general results of this paper hold for these classes and their subclasses.

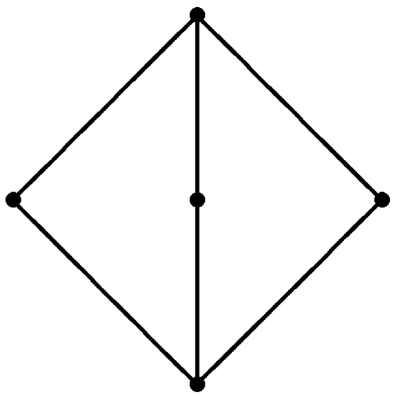

Fig. 1. $K_{2,3}$.

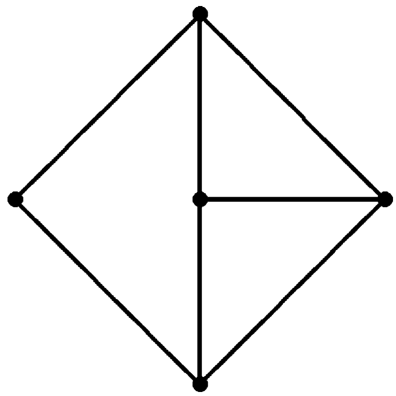

Fig. 2.

In particular, we established in [10] that the elementary weakly median graphs are 5-wheels, complete graphs possibly minus a matching and 2-connected weakly median bridged graphs (see [2,10]). Since it is straightforward to check that every complete graph possibly minus a matching and every wheel are moorable graphs and by Proposition 2.1.3, we deduce the following proposition.

2.2.1. Proposition. *Every elementary weakly median graph is moorable.*

2.3. Retractions of Cartesian products

For any Cartesian product of connected graphs with at least one edge $H = \Box_{i\in I} H_i$, for any $a \in V(H)$ and for any $j \in I$, we defined in [10, Section 2.2] the subgraph $H(j,a)$ as the fiber of H which is the copy of H_j containing a. Recall that pr_j is the jth coordinate function of the Cartesian product $\prod_{i\in I} V(H_i)$ onto $V(H_j)$.

2.3.1. Theorem. *Let $H := H_0 \Box H_1$ where H_0 and H_1 are connected graphs and let a be some vertex of H. For $i \in \{0,1\}$, let $G_i := H(i,a)$, and put $G := G_0 \cup G_1$. Then G is a retract of H if and only if there exist moorings from H_0 and H_1 onto $\{\mathrm{pr}_0(a)\}$ and $\{\mathrm{pr}_1(a)\}$, respectively.*

Proof. With the notation of the lemma, suppose that there exists a retraction $\psi : H \to G$ and that H_1 is not trivial (otherwise, we are done). Let $a' \in V(G_1)$ be some neighbor of a and put $G_0' := H(0,a')$. For every $x \in V(H_0)$, let y be the vertex of H such that $\mathrm{pr}_0(y) = x$ and $\mathrm{pr}_1(y) = \mathrm{pr}_1(a)$ and let y' be the neighbor of y in G_0'. Then it is straightforward to check that $y' \in I_H(y,a')$; thus $\psi(y') \in I_H(\psi(y),\psi(a')) = I_H(y,a')$. It results that $\psi(y') \in I_{G_0}(y,a)$. Put $x' := \mathrm{pr}_0(\psi(y'))$. Since $x' \in I_{H_0}(\mathrm{pr}_0(y),\mathrm{pr}_0(a)) = I_{H_0}(x,\mathrm{pr}_0(a))$ and $d_{H_0}(x,x') = 1$, we define a mooring φ_0 of H_0 onto $\{\mathrm{pr}_0(a)\}$ by $\varphi_0(x) := x'$. We prove the existence of a mooring of H_1 onto $\{\mathrm{pr}_1(a)\}$ in the same way.

Conversely, for $i \in \{0,1\}$, let φ_i be a mooring of H_i onto its prefiber $\{\mathrm{pr}_i(a)\}$ and for every $x \in V(H)$, let $n_i := d_{H_i}\ (\mathrm{pr}_i(x), \mathrm{pr}_i(a))$. Let ψ be the map of $V(H) = V(H_0) \times V(H_1)$ into $V(G)$ defined by

$$\psi(x) = (\varphi_0^{n_1}(\mathrm{pr}_0(x)), \mathrm{pr}_1(a)) \quad \text{or} \quad (\mathrm{pr}_0(a), \varphi_1^{n_0}(\mathrm{pr}_1(x))),$$

according to whether $n_0 \geqslant n_1$ or not.

We will show that ψ is an endomorphism. Let $\{x, x'\}$ be an edge of H with $n_i := d_{H_i}\ (\mathrm{pr}_i(x), \mathrm{pr}_i(a))$ and $n_i' := d_{H_i}\ (\mathrm{pr}_i(x'), \mathrm{pr}_i(a))$ for $i \in \{0,1\}$. Since $d_H(x,a) = n_0 + n_1$ and $d_H(x',a) = n_0' + n_1'$, without loss of generality, we can suppose that $\{x, x'\}$ is an edge of a copy of H_0 in H with $n_0 \leqslant n_0'$. Thus $\mathrm{pr}_1(x) = \mathrm{pr}_1(x')$, $n_1 = n_1'$ and $0 \leqslant n_0' - n_0 \leqslant 1$, and three cases are to be considered.

1. $n_0 = n_0'$.
   Let i be the least integer in $\{0,1\}$ such that $n_i \geqslant n_{1-i}$. $\mathrm{pr}_i(\psi(x))$ and $\mathrm{pr}_i(\psi(x'))$ are the images of x and x' by the same contraction $\varphi_i^{n_{1-i}} \circ \mathrm{pr}_i$; thus $\{\mathrm{pr}_i(\psi(x)), \mathrm{pr}_i(\psi(x'))\}$ is an edge of H_i or $\mathrm{pr}_i(\psi(x)) = \mathrm{pr}_i(\psi(x'))$ and $\{\psi(x), \psi(x')\}$ is an edge of G or $\psi(x) = \psi(x')$.
2. $n_0 = n_0' - 1$ and $n_0 \geqslant n_1$.
   $\mathrm{pr}_0(\psi(x))$ and $\mathrm{pr}_0(\psi(x'))$ are the images of x and x' by the same contraction $\varphi_0^{n_1} \circ \mathrm{pr}_0$; thus $\{\psi(x), \psi(x')\}$ is an edge of G or $\psi(x) = \psi(x')$.
3. $n_0 = n_0' - 1$ and $n_0 < n_1$.
   In this case, $\mathrm{pr}_1(\psi(x')) = \varphi_1^{n_0'} \circ \mathrm{pr}_1(x') = \varphi_1^{n_0+1} \circ \mathrm{pr}_1(x) = \varphi_1\ (\mathrm{pr}_1(\psi(x)))$; thus $\{\mathrm{pr}_1(\psi(x)), \mathrm{pr}_1(\psi(x'))\}$ is an edge of H_1 by Definition 2.1.1 since φ_1 is a mooring, and $\{\psi(x), \psi(x')\}$ is an edge of G as well.

Hence ψ maps the edge $\{x, x'\}$ to an edge or a vertex of G; thus it is an endomorphism. Since the restriction of ψ to G is the identity map of G, ψ is a retraction of H onto G. $\square$

2.4. Generalized amalgamations and expansions

2.4.1. Definition.

1. *Generalized amalgamation.* Let U be a prefiber of a graph G. If $(W_i)_{i \in I}$ is a family of prefibers of G with $\bigcup_{i \in I} W_i = V(G)$ and, for every $i \neq j$, $W_i \cap W_j = U$, then G is the *generalized gated amalgam* (or simply the *amalgam*) of the family $(G[W_i])_{i \in I}$.
2. *Generalized expansion.* Let G be a graph which is the generalized gated amalgam of a family $(W_i)_{i \in I}$ of prefibers, with $\bigcap_{i \in I} W_i = U$. If C is an elementary graph for which there exists a bijection $\kappa : I \to V(C)$, then a *generalized gated expansion* (or simply an *expansion*) of G with respect to C and $(W_i)_{i \in I}$ is the subgraph of $G \square C$ induced by $\bigcup_{i \in I} (W_i \times \{\kappa(i)\})$ (that is, the disjoint union of copies of the graphs induced by every W_i where the copies of U induce a subgraph isomorphic to $G[U] \square C$).
3. *Contraction.* Let G be a fiber-complemented graph and let S be some elementary prefiber of G. For any a and b in S, we denote by φ^{ab} the isomorphism between

$U_S(a)$ and $U_S(b)$; let γ^S be the equivalence relation on $V(G)$ defined by

$$x\gamma^S y \Leftrightarrow y = \varphi^{ab}(x) \quad \text{for some } a, b \text{ in } S.$$

The classes of γ^S are the elementary prefibers parallel to S and the one element sets $\{x\}$ for every x belonging to $W_S(a) - U_S(a)$ with $a \in S$. The *contraction of G with respect to S* is the graph G' whose vertices are the γ^S-classes, and for every distinct elements x' and y' of $V(G')$, $\{x', y'\}$ is an edge of G' if and only if there exist $x \in x'$ and $y \in y'$ with $\{x, y\} \in E(G)$ (roughly speaking, G' is the graph deduced from G by contracting each γ^S-class into a single vertex).

2.4.2. Lemma.

1. *If G is the amalgam of the family $(W_i)_{i \in I}$ of prefibers, where each prefiber W_i induces a fiber-complemented graph, then G is a fiber-complemented graph.*
2. *If H is the expansion of a fiber-complemented graph G with respect to some elementary graph C and to some family $(W_i)_{i \in I}$ of prefibers of G, then H is a fiber-complemented graph.*
3. *If G' is the contraction of a fiber-complemented graph G with respect to some elementary prefiber S, then G' is a fiber-complemented graph.*

Proof. (1) One can easily extend the proof of Lemma 5.7 in [10] to generalized amalgams to obtain the first part of the lemma.

(2) If G is a fiber-complemented graph, then each W_i induces a fiber-complemented graph by [10, Lemma 4.4]. Let H_i be the subgraphs of H induced by the copies of W_i and let K by the subgraph induced in H by the copies of U in each H_i. It is straightforward to check that, for every $i \in I$, $H_i \cup K$ induces a fiber-complemented graph, whose vertex set is a prefiber of H. Thus, H is the generalized amalgam of $(V(H_i \cup K))_{i \in I}$ and is a fiber-complemented graph by the first part of the lemma.

(3) Clearly, G' is the amalgam of the family $(W_S(x)/\gamma^S)_{x \in S}$. $\square$

3. Retractions

3.1. Prefibers and retractions

First, notice that if $\rho : H \to G$ is a retraction of a connected graph H onto a subgraph G, then obviously G is an isometric subgraph of H.

The following lemma due to Tardif [25] was established for finite metric spaces; however his proof is valid for graphs by using the discrete feature of their structure of metric space.

3.1.1. Lemma (Tardif [25, Lemma 4.4]). *If G is a retract of a connected graph H and if W is a prefiber of H, then $W \cap H$ is a prefiber of G if it is nonempty.*

3.2. The main theorem

We recall a notation introduced in [10, Section 6.2]: for any fiber-complemented graph G, $\mathscr{S}_G$ denotes the set of the classes of the parallelism relation between elementary prefibers of G, and each element of $\mathscr{S}_G$ is identified with a graph induced by some of its representatives.

3.2.1. Theorem. *Let G be a fiber-complemented graph whose elementary prefibers induce moorable graphs. Then G is a retract of the Cartesian product $H = \square_{C \in \mathscr{S}_G} C$.*

Proof. For any graph F, the notation proj_W^F designates the projection map of $V(F)$ onto some prefiber W of F, and for $A \subseteq V(F)$, $\mathrm{pref}_F(A)$ is the prefiber of F generated by A.

By Chastand [10, Theorem 6.1], G is identified with an isometric subgraph of $H = \square_{C \in \mathscr{S}_G} C$, where every $C \in \mathscr{S}_G$ is an elementary graph isomorphic to the subgraph induced by some elementary prefiber of G. Thus if every element of $\mathscr{S}_G$ is moorable, then H is also a fiber-complemented graph whose elementary prefibers induce moorable graphs, whereas, for every elementary prefiber S of G, we will put $W_S(x) := (\mathrm{proj}_S^G)^{-1}(x)$ and we will denote by $U_S(x)$ the set of vertices of $W_S(x)$ which have a neighbour in $G - W_S(x)$ ($W_S(x)$ and $U_S(x)$ are prefibers of G by Chastand [10, Theorem 5.2]).

The proof of the theorem requires several steps.

1. Construction of a fiber-complemented extension G_1 of G.
   Suppose that G is not a Cartesian product of moorable elementary graphs (otherwise, we are done). By Chastand [10, Theorem 6.6] there exists in G an induced path $\langle u, b, v \rangle$ of length 2 such that v does not belong to $S := \mathrm{pref}_G(\{b, u\})$; put $T := \mathrm{pref}_G(\{b, v\})$. There exists $a \in V(H) - V(G)$ adjacent to u and v but not to b since S and T are Δ-closed by p-convexity. Furthermore S and T induce copies of some factors C_S and C_T in $\mathscr{S}_G$. We denote by P the subgraph of H isomorphic to $G[S] \square G[T]$, which is the unique copy of $C_S \square C_T$ containing S and T. Note that $T - \{b\} \subseteq W_S(b) - U_S(b)$ since if an element of $T - \{b\}$ belonged to $U_S(b)$, G would contain P and consequently the above defined vertex a.
   Put $U := U_S(b) \cap U_T(b)$. Let X be the subgraph of H isomorphic to $P \square G[U]$ which contains P and U. We define the extension G_1 of G as the subgraph of H induced by $G \cup X$.
   By construction, G_1 is isomorphic to the graph G'' obtained from G by the two following processes:
   (a) Let G' be the contraction of G with respect to the elementary prefiber S.
   Let γ^S be the equivalence relation associated to the contraction of G (see Definition 2.4.1). We denote by y' the γ^S-class of some $y \in V(G)$ and by W' the quotient set of some subset W of $V(G)$. Notice that, for every $x \in S$, the class of $y \in W_S(x) - U_S(x)$ is a one element set, $W_S'(x) \approx W_S(x)$ and $U_S'(x) = U_S'(z)$ for every $z \in S$; put $U_S' := U_S'(x)$.
   G' is the amalgam of the family $(W_S'(x))_{x \in S}$, with $U_S' := \bigcap_{x \in S} W_S'(x)$. Hence, G' is a fiber complemented graph by Lemma 2.4.2 since every $W_S'(x)$ induces a

fiber-complemented graph. Thus $T' = T/\gamma^S$ is an elementary prefiber of G', with $(\mathrm{proj}_{T'}^{G'})^{-1}(y') = W'_T(y) \approx W_T(y)$ for $y' \in T' - b'$, while $(\mathrm{proj}_{T'}^{G'})^{-1}(b') = W'_T(b)$ is the amalgam of $(W_S(b) \cap W_T(b))/\gamma^S$ and the sets $W'_S(x)$ for every $x \in S - \{b\}$, whose intersection set is U'_S.
Put $B(b') := U' = U'_S(b) \cap U'_T(b)$ (recall that $U := U_S(b) \cap U_T(b)$) and $B(y') := \mathrm{proj}_{W'_T(y)}^{G'}(U')$ for every $y' \in T' - \{b'\}$. Clearly, the set

$$A := U'_S(b) \cup \left(\bigcup_{y' \in T' - \{b'\}} B(y') \right)$$

is a prefiber of G' since there are no edges between $U'_S(b) - B(b')$ and $W'_T(y) - B(y')$ for every $y' \in T' - \{b'\}$.
Now, consider the family $\mathscr{F} := (D(x))_{x \in S}$ of subsets of $V(G')$ consisting on the one hand of the prefiber

$$D(b) := (W_S(b) \cap W_T(b))/\gamma^S \cup \left(\bigcup_{y' \in T' - \{b'\}} W'_T(y) \right)$$

and on the other hand of the prefibers

$$D(x) := W'_S(x) \cup \left(\bigcup_{y' \in T' - \{b'\}} B(y') \right) \quad \text{for every } x \in S - \{b\}.$$

Its intersection is equal to A and there are no edges between $D(x) - A$ and $D(z) - A$ for $x, z \in S$. Thus G' is the amalgam of the family $\mathscr{F}$.

(b) Let G'' be the graph obtained from G' by the expansion with respect to the family $\mathscr{F}$ and the elementary graph induced by S.
By Lemma 2.4.2, G'' is a fiber-complemented graph and so is G_1. Moreover, obviously the elementary prefibers of G_1 induce moorable graphs and G_1 embeds isometrically in H.

2. **Claim.** *There exists a retraction $\rho_1 : G_1 \to G$.*
Proof. By Theorem 2.3.1, there exists a retraction $r : P \to G[S \cup T]$ since, by hypothesis, every elementary prefiber of G induces a moorable graph. The isomorphism $\varphi : X \to P \,\Box\, G[U]$ enables us to extend the retraction r to a retraction $r' : X \to X \cap G$ by $r'(x) := \varphi^{-1}(r(\mathrm{pr}_P(\varphi(x)), \mathrm{pr}_U(\varphi(x)))$ for every $x \in X$, where pr_P and pr_U are the coordinate functions of $V(P) \times U$ onto $V(P)$ and U, respectively. Finally for every $x \in V(G_1)$, put $\rho_1(x) := r'(x)$ or x according to whether x belongs to X or not. □
3. Construction of an increasing sequence of retracts.
Put $G_0 := G$ and $\zeta_1 := \rho_1$. We will construct an increasing sequence $(G_\alpha)_{\alpha \geqslant 0}$ of fiber-complemented graphs whose elementary prefibers induce moorable graphs and a sequence $(\zeta_\alpha)_{\alpha \geqslant 1}$ of retractions with $\zeta_\alpha : G_\alpha \to G_0$.
Suppose that G_α and ζ_α have been already defined. If G_α is different from H (that is, if G_α is not a Cartesian product), then there exist an isometric embedding of G_α into H and a vertex $a_\alpha \in V(H - G_\alpha)$ which enables us to build with the above

described process an extension $G_{\alpha+1}$ and a retraction $\rho_{\alpha+1}: G_{\alpha+1} \to G_\alpha$. Clearly, $G_{\alpha+1}$ is a fiber-complemented graph by construction and the map $\zeta_{\alpha+1} := \rho_{\alpha+1} \circ \zeta_\alpha$ is a retraction from $G_{\alpha+1}$ onto G_0 by Claim 2.

If α is a limit ordinal, let G_α be the subgraph of H induced by $\bigcup_{\beta<\alpha} G_\beta$; the map $\zeta_\alpha : G_\alpha \to G_0$ is defined by $\zeta_\alpha := \bigcup_{\beta<\alpha} \zeta_\beta$. Note that, for every $\gamma<\beta<\alpha$, the domain of ζ_γ is contained in the domain of ζ_β and ζ_γ is the restriction of ζ_β to G_γ; hence ζ_α is a retraction.

4. **Claim.** *If α is a limit ordinal and if G_α is the subgraph of H induced by $\bigcup_{\beta<\alpha} G_\beta$, then G_α is a fiber-complemented graph whose elementary prefibers induce moorable graphs.*

Proof. Let W be a prefiber of G_α and let $x \in W$. Put $U := (\mathrm{proj}_W^{G_\alpha})^{-1}(x)$; we will prove that U is a prefiber of G_α. Let η be the least ordinal β with $x \in G_\beta$.

First, clearly, G_α is an isometric subgraph of H, and, for every $\beta<\alpha$, the map $\bigcup_{\beta<\gamma<\alpha} \zeta_\gamma$ is a retraction of G_α onto G_β. Thus by Lemma 3.1.1, for every $\beta<\alpha$, $W_\beta := W \cap G_\beta$ is a prefiber of G_β if it is nonempty and $V_\beta := (\mathrm{proj}_{W_\beta}^{G_\beta})^{-1}(x)$ is a prefiber of G_β for $\eta \leqslant \beta < \alpha$ since G_β is fiber-complemented (otherwise we put $V_\beta := \emptyset$ for $\beta<\eta$). Furthermore, for $\beta<\gamma$ and for $y \in G_\beta$, $\mathrm{proj}_{W_\beta}^{G_\beta}(y)=x$ implies $\mathrm{proj}_{W_\gamma}^{G_\gamma}(y)=x$ since G_β is an isometric subgraph of G_γ. It results that $(V_\beta)_{\beta<\alpha}$ is an increasing sequence of nested subsets of $V(G_\alpha)$.

Besides, let $y \in U$ (that is, $\mathrm{proj}_W^{G_\alpha}(y)=x$). Let β such that $x, y \in G_\beta$ and put $\mathrm{proj}_{W_\beta}^{G_\beta}(y)=z$. Since G_β is an isomorphic subgraph of G_α and W_β is a prefiber of G_β, $d(y, W_\beta)=d(y,z) \leqslant d(y,x)$, and since W is a prefiber of G_α, $d(y,z) \geqslant d(y,W) = d(y,x)$. Thus $z=x$, $y \in V_\beta$ and $U \subseteq V$.

Conversely, let $y \in V$ and put $z := \mathrm{proj}_W^{G_\alpha}(y)$. There exists γ such that $\mathrm{proj}_{W_\beta}^{G_\beta}(y)=x$ and $z \in W_\gamma$ for every β with $\gamma \leqslant \beta < \alpha$; since W_β is a prefiber of G_β, $d(y,z) \geqslant d(y, W_\beta) = d(y,x)$, and since W is a prefiber of G_α, $d(y,x) \geqslant d(y,W) = d(y,z)$. Thus $z=x$, $\mathrm{proj}_W^{G_\alpha}(y)=x$ and $V \subseteq U$. Consequently, $U=V$.

Moreover, let $z \in V(G_\alpha)$ and $t \in V$. We will prove that there exists $y \in V$ such that $y \in I_{G_\alpha}(z,t)$. There exists γ such that $x,z,t \in V(G_\beta)$ for every β with $\gamma \leqslant \beta < \alpha$ and we construct a sequence $(y_\beta)_{\gamma \leqslant \beta < \alpha}$ of vertices of G_α such that $y_\beta := \mathrm{proj}_{W_\beta}^{G_\beta}(z)$. Since G_β is an isometric subgraph of G_α and of $G_{\beta'}$ for $\beta<\beta'$, and since the V_β's are nested, the sequence $(d(z, y_\beta))_{\gamma \leqslant \beta < \alpha}$ is decreasing. Let β and β' be two ordinals with $\gamma \leqslant \beta < \beta' < \alpha$ and such that $d(z, y_\beta) = d(z, y_{\beta'})$. The vertex y_β is an element of V_β which is a subset of the prefiber $V_{\beta'}$ of $G_{\beta'}$; hence $y_\beta = y_{\beta'}$. Thus the sequence $(y_\beta)_{\gamma \leqslant \beta < \alpha}$ is finite and there exists β_0 such that $y_\beta = y_{\beta_0}$ for $\beta_0 \leqslant \beta < \alpha$ and such that $y_{\beta_0} \in I_{G_\alpha}(z,t)$ for every $t \in V$, which prove that U is a prefiber of G_α.

Consequently, G_α is a fiber-complemented graph, whose elementary prefibers induce moorable graphs by construction. □

Therefore, there exists an ordinal α such that $G_\alpha = H$ since we construct an increasing sequence of subgraphs of H such that every vertex of H belongs to a subgraph of this sequence. Finally ζ_α is a retraction of H onto $G_0 = G$, which completes the proof. □

3.3. Moorings onto prefibers

3.3.1. Theorem. *Let G be a fiber-complemented graph. The following assertions are equivalent*:

(i) *G is moorable*;
(ii) *every elementary prefiber of G induce a moorable graph*;
(iii) *G admits moorings onto every prefiber.*

Proof. (iii) $\Rightarrow$ (i) is obvious and (i) $\Rightarrow$ (ii) is immediate since, for every vertices x and y belonging to the same elementary prefiber S, $I_G(x,y)$ is contained in S. It remains to prove (ii) $\Rightarrow$ (iii).

Let G be a fiber-complemented graph whose elementary prefibers induce moorable graphs, and let W be a prefiber of G. Consider the subgraph G' of the graph $L := G \Box K_2$ induced by $(V(G)\times\{0\})\cup(W\times\{1\})$, where $\{0,1\}$ is the vertex set of the graph K_2. L is a fiber-complemented graph as a Cartesian product of fiber-complemented graphs by [10, Theorem 4.5]. In the following we identify G and the fiber which is the subgraph of L induced by $V(G)\times\{0\}$.

This graph G' is the expansion of G with respect to its prefibers $V(G)$ and W and the elementary graph K_2. However, since W is a prefiber of G, $(W\times\{0\})\cup(W\times\{1\})$ is a prefiber of L by Tardif [23, Lemma 3.1]; thus G' is the amalgam of two prefibers $V(G)\times\{0\}$ and $(W\times\{0\})\cup(W\times\{1\})$ as well; hence G' is a fiber-complemented graph.

Claim. *There exists a retraction $\rho : G\Box K_2 \to G'$.*

Proof. The proof of this claim is quite similar to the proof of Theorem 3.2.1 and we only give the main steps. The graphs G, G' and L are isometric subgraphs of the graph $H = \Box_{C\in S_L} C$ where $\mathscr{S}_L$ is the set of the classes of the parallelism relation between the elementary prefibers of L. We construct a sequence $(G'_\alpha)_{\alpha\geqslant 0}$ of subgraphs of L which are fiber-complemented and a sequence of retractions $(\zeta_\alpha)_{\alpha\geqslant 1}$ with $\zeta_\alpha : G'_\alpha \to G'_0$, as follows.

Put $G'_0 := G'$ and $\zeta_0 := Id_G$. Suppose that G'_α and ζ_α have already been constructed. If G'_α is different from $L = G\Box K_2$, then there exists a vertex $b\in V(G'_\alpha)$ adjacent to some $u\in V(G)$ and some $v\in V(G'_\alpha - G)$ such that the path $\langle u,b,v\rangle$ is not g-convex in H. Hence, as described in the proof of Theorem 3.2.1, there exists some vertex $a\in V(H)-V(G)$ with which we construct an extension $G'_{\alpha+1}$ of G'_α and a retraction $\rho_{\alpha+1} : G'_{\alpha+1}\to G'_\alpha$ since all elementary prefibers of G' induce moorable graphs. Put $\zeta_{\alpha+1} := \rho_{\alpha+1}\circ\zeta_\alpha$. By construction, this vertex a belongs to $V(L)$; thus $G'_{\alpha+1}$ is an isometric subgraph of L and it is straightforward to check that it is also a fiber-complemented graph.

If α is a limit ordinal, we put $G'_\alpha := \bigcup_{\beta<\alpha} G'_\beta$ and $\zeta_\alpha := \bigcup_{\beta<\alpha}\zeta_\beta$. Since G'_β is a fiber-complemented graph contained in L for every $\beta<\alpha$, so is G'_α.

Finally, there exists an ordinal α with $G'_\alpha = L$ and the map $\rho := \zeta_\alpha$ is a retraction of L onto G'. $\Box$

Let $\varphi: V(G) \to V(G)$ defined for every $u \in V(G)$ by $\varphi(u) := \mathrm{pr}_G(\rho(u,1))$, where pr_G is the projection map of $V(L)$ onto its factor $V(G)$. Obviously, φ is an endomorphism of G and we will show that it is a mooring.

If u belongs to W, then $(u,1) \in V(G')$; thus $\varphi(u) = \mathrm{pr}_G(\rho(u,1)) = \mathrm{pr}_G(u,1) = u$.

If u belongs to $V(G) - W$, then $\rho(u,1) = (\varphi(u),0)$ and $d_G(u,\varphi(u)) \leqslant 1$. Put $u' := \mathrm{proj}_W(u)$; thus $d_L(\rho(u,1),\rho(u',1)) = d_L((\varphi(u),0),(u',1))$ and $d_L(\rho(u,1),\rho(u',1)) \leqslant d_L((u,1),(u',1)) = d_G(u,u')$. Otherwise, $V(G) \times \{0\}$ is a prefiber of G' such that the projection of $(u',1)$ is $(u',0)$; thus $d_L((u',1),(\varphi(u),0)) = d_L((u',0),(\varphi(u),0)) - 1 = d_G(u',\varphi(u)) - 1$. Hence $d_G(u',\varphi(u)) \leqslant d_G(u,u') + 1$ and it results that $\{u,\varphi(u)\}$ is an edge of $G[I_G(u,u')]$, which proves that φ is a mooring of G onto W. $\square$

3.4. Retractions of pre-median graphs

Since every elementary weakly median graph is moorable by Proposition 2.2.1 and every bridged graph is moorable by Proposition 2.1.3, Theorem 3.2.1 admits the immediate following corollary related to some subclasses of the class of moorable pre-median graphs.

3.4.1. Corollary.

1. *Every median graph is a retract of an hypercube* [1, *Theorem* 2].
2. *Every quasi-median graph is a retract of a Hamming graph* [9, *Theorem* 4.1].
3. *Every pseudo-median graph is a retract of a Cartesian product of elementary pseudo-median graphs* (*note that this Cartesian product is not in general a pseudo-median graph*).
4. *Every weakly median graph is a retract of a Cartesian product of elementary weakly median graphs.*
5. *Every bridged graph is a retract of the Cartesian product of its blocks* [11, *Theorem* 4.6].

However, the retract of a Cartesian product of (moorable) elementary fiber-complemented graphs is not necessarily a fiber-complemented graph. In order to assure that such a retract is fiber-complemented, we are led in the following sections to restrict our study to pre-median graphs and to introduce the concept of pre-median subgraph. We will use the basic notations already defined in [10, Section 4.3].

3.4.2. Definition. A connected induced subgraph H of a pre-median graph G is a *pre-median subgraph* of G if every triple of vertices of H satisfying the hypothesis of the $\diamond$ and ∇-properties has a quasi-median (or a median) in H. More precisely, H is a pre-median subgraph of G if it satisfies both following properties ($\diamond$ and ∇-properties for pre-median subgraphs):

1. for every vertices u,v,w in H with $d_G(v,w) = 1$ and $d_G(u,v) = d_G(u,w) = k > 1$, there exists a common neighbor x of v and w in $V(H)$ with $d_G(u,x) = k - 1$.

2. for every vertices u, v, w, z in H with $d_G(v, w) = 2$, $d_G(u, v) = d_G(u, w) = d_G(u, z) - 1 = k > 1$ and z a common neighbor of v and w, there exists a common neighbor x of v and w in $V(H)$ with $d_G(u, x) = k - 1$.

Note that the vertex x in these above statements is a median or belongs to the quasi-median of the triple (u, v, w). Thus it is easy to check that one can define equivalently a pre-median subgraph by the following:

A connected induced subgraph H of a pre-median graph G is a pre-median subgraph of G if and only if at least one of the quasi-medians or the medians of every triple of vertices of H is contained in $V(H)$.

3.4.3. Lemma. *If H is a pre-median subgraph of a pre-median graph G, then H is an isometric subgraph of G (that is, d_G and d_H coincide on $V(H)$).*

Proof. Suppose that H is not isometric. Let u and v be in $V(H)$ with $d_H(u, v) > d_G(u, v)$ and suppose that u and v are chosen such that $k = d_H(u, v)$ is minimal among the ordered pairs (u, v) with $d_H(u, v) > d_G(u, v)$ and such that $d_G(u, v)$ is minimal among such ordered pairs (u, v). Note that necessarily $d_G(u, v) \geqslant 3$.

Let $w \in I_H(u, v) \cap N(v)$. The choice of (u, v) implies that $d_H(u, w) = d_G(u, w) = k - 1$; thus $k - 2 \leqslant d_G(u, v) \leqslant k - 1$ and two cases are to be examined.

1. If $d_G(u, v) = k - 1$, then by the ∇-property of Definition 3.4.2, there exists $x \in V(H)$ adjacent to v and w with $d_G(u, x) = k - 2$. Since x is adjacent to v, then $k - 1 \leqslant d_H(u, x) \leqslant k + 1$ and since x is adjacent to w, then $k - 2 \leqslant d_H(u, x) \leqslant k$. Thus $k - 1 = d_G(u, x) < k - 1 \leqslant d_H(u, x) \leqslant k$, which is conflicting with the choice of (u, v).
2. If $d_G(u, v) = k - 2$, then let $t \in I_H(u, w) \cap N(w)$. The minimality of k implies that $d_G(u, t) = d_H(u, t) = k - 2$ and by the $\diamond$-property of Definition 3.4.2, there exists $x \in V(H)$ adjacent to v and t with $d_G(u, x) = k - 3$. Since x is adjacent to v, then $k - 1 \leqslant d_H(u, x) \leqslant k + 1$ and since x is adjacent to t, then $k - 2 \leqslant d_H(u, x) \leqslant k$. Thus $k - 3 = d_G(u, x) < k - 1 \leqslant d_H(u, x) \leqslant k$, which is conflicting with the choice of (u, v).

It results that there exists no ordered pair (u, v) of elements of $V(H)$ with $d_H(u, v) > d_G(u, v)$ and H is an isometric subgraph of G. $\square$

Obviously a pre-median subgraph of a pre-median graph cannot contain the graph of Fig. 1 or 2 as an induced subgraph; thus we deduce the corollary:

3.4.4. Corollary. *Every pre-median subgraph of a pre-median graph is a pre-median graph.*

Recall that if $\rho : H \to G$ is a retraction of a connected graph H onto a subgraph G, then G is an isometric subgraph of H. Moreover, whenever H is a pre-median graph, we have a more precise information:

3.4.5. Lemma. *If $\rho : H \to G$ is a retraction of a pre-median graph H onto a subgraph G, then G is a pre-median subgraph of H.*

Proof. Since G is an isometric subgraph, then $\rho(I_H(u,v)) \subseteq I_G(u,v)$ for every vertices u and v of G. Thus the $\diamond$ and ∇-properties hold in G, which implies that G is a pre-median subgraph of H. $\square$

By Chastand [10, Theorem 4.5], every Cartesian product of pre-median graphs is pre-median. Thus it follows:

3.4.6. Corollary. *Every retract of a Cartesian product of elementary pre-median graphs is a pre-median graph.*

4. Endomorphisms of a moorable pre-median graph

4.1. Vertices of finite period under an endomorphism

In the following, we consider only graphs with finite elementary prefibers. From Theorem 3.2.1, we deduce a property related to the polytopes of such graphs.

4.1.1. Proposition. *If G is a moorable fiber-complemented graph without infinite elementary prefibers, then the prefiber generated by any finite family $\mathscr{F}$ of vertices of G is finite (that is, the p-convex hull of $\mathscr{F}$ is finite, and its g-convex hull as well).*

Proof. Let F be a finite subset of $V(G)$, where G is a moorable fiber-complemented graph whose elementary prefibers are finite. Obviously, the elementary box B of $H = \square_{C \in \mathscr{S}_G} C$ generated by F is finite. By Theorem 3.2.1 and Lemma 3.1.1, $B \cap G$ is a prefiber of G since it is nonempty. $\square$

It results in particular that every moorable fiber-complemented graph without infinite elementary prefibers is interval-finite (that is, every interval is finite).

4.1.2. Notation. Let f be an endomorphism of a graph G. We denote by G_f the subgraph of G induced by the set of the vertices of G of finite period under f, that is,

$$G_f := G[\{u \in V(G):\ \text{there exists an integer } n \geqslant 0 \text{ with } f^n(u) = u\}].$$

4.1.3. Lemma. *If G is an interval finite pre-median graph and if f is an endomorphism of G, then*

(a) *G_f is a pre-median subgraph of G;*
(b) *the restriction of f to G_f is an automorphism of G_f;*
(c) *moreover, if G contains no isometric ray and no infinite elementary prefibers (that is, if G is compact according to Chastand [10, Theorem 7.10]) and if G_f is nonempty, then G_f contains a finite box S which is strictly stabilized by f (that is, $f(S) = S$).*

Proof. (a) Suppose that G_f is nonempty. Let u and v be two vertices of G_f. Since G is interval-finite, there exist finitely many uv-geodesics. Thus at least one of these geodesics is of finite period under f. Hence each vertex of this geodesic belongs to G_f and G_f is a connected subgraph.

Let u, v, w be three vertices of G_f satisfying the hypothesis of the $\diamond$ or the ∇-property, with $d_G(u,v) = d_G(u,w) = k$. Since G is interval-finite, there exist in $V(G)$ only finitely many vertices x adjacent to v and w with $d_G(u,x) = k-1$. Necessarily, one of these vertices is of finite period which is also a common multiple of the periods of u, v and w. Thus this vertex belongs to $V(G_f)$. It results that G_f is a pre-median subgraph of G by Definition 3.4.2.

(b) Clearly, $f(G_f) \subseteq G_f$ and the restriction of f to $V(G_f)$ is a bijection. Moreover, $\{f(u), f(v)\}$ belongs to $E(G_f)$ if and only if $\{u,v\} \in E(G_f)$. Thus the restriction of f to $V(G_f)$ is an automorphism of G_f.

(c) This is a consequence of Chastand [10, Theorem 8.8]. Note that the elementary prefibers of G_j do not necessarily coincide with elementary prefibers of G; however, every elementary prefiber of G_f is contained in some elementary prefiber of G. □

4.2. Endomorphisms and compacticity

In [10, Section 7], we endowed the vertex set of a fiber-complemented graph G with the topology (denoted by $\mathscr{T}$) for which a subbase is (in terms of closed sets) the family of all copoints of G, and we showed in [10, Theorem 7.10] that this topological space $(V(G), \mathscr{T})$ is a compact Hausdorff space if and only if G contains no isometric rays and no infinite elementary prefibers.

4.2.1. Theorem. *Let G be a compact Hausdorff moorable pre-median graph and let f be an endomorphism of G. Then $V(G)$ contains a finite nonempty subset S which is strictly stabilized by f (that is, such that $f(S) = S$).*

Proof. The proof is by induction on $\delta_f := \min\{d_G(u, f(u)) \colon u \in V(G)\}$. The result is obvious if $\delta_f = 0$ and we suppose that it is true for every endomorphism g of any compact Hausdorff moorable pre-median graph with $\delta_g \leqslant \delta_f$.

Following Tardif [24], a subset S of $V(G)$ is called a *quasi-invariant* subset by f if, for every $u \in S$, either $f(u) \in S$ or there exists $v \in S$ with $\{f(u), v\} \in E(G)$. We make two remarks.

(a) If S is quasi-invariant by f, then its successive images by f are also quasi-invariant.
(b) For every prefiber W of G, a subset S of W is quasi-invariant by f whenever $\varphi \circ f(S) \subseteq S$ for every mooring φ of G onto W, since φ coincides with proj_W for all vertices of G at distance at most 1 from W.

Claim. *G contains a finite box B which is quasi-invariant under f.*

Proof. Let $u \in V(G)$ with $d_G(u, f(u)) = \delta_f$ and let φ be a mooring of G onto $\{u\}$. Thus $d_G(u, \varphi \circ f(u)) = \delta_f - 1$ and by the induction hypothesis, there exists a finite nonempty set $S \subseteq V(G)$ such that $\varphi \circ f(S) = S$. It results that $G_{\varphi\circ f}$ is nonempty and by Lemma 4.1.3 there exists a finite nonempty box B with $\varphi \circ f(B) = B$, whose elementary prefibers are pre-median subgraphs of some elementary prefibers of G. Hence B is a box which is quasi-invariant under f. $\square$

Since G is compact Hausdorff, its elementary prefibers are finite by Chastand [10, Theorem 7.10] and, for every above defined box B, the prefiber generated by B is finite by Proposition 4.1.1.

Let S_0 be a quasi-invariant subset of $V(G)$ of minimal cardinality such that $\sum_{x,y\in S_0} d_G(x,y)$ is minimal (note that necessarily S_0 induces a box of G). We define by induction the sequence $(S_n)_{n\geqslant 0}$ of subsets of $V(G)$ and the sequence $(B_n)_{n\geqslant 0}$ of prefibers of G by putting $S_{n+1} := f(S_n)$ and $B_n := \mathrm{pref}(S_n)$ for every $n \geqslant 0$. Since every S_n is finite, every B_n is a finite box of G. Let φ_n be a mooring of G onto B_n. We will describe the sequences $(S_n)_{n\geqslant 0}$ and $(B_n)_{n\geqslant 0}$ in several steps. First, by the above remark, every S_n is quasi-invariant under f. Thus, $|S_n| = |S_0|$ since the inequality $|S_n| < |S_0|$ is conflicting with the minimality of $|S_0|$. Moreover, S_n is the image of S_0 by the endomorphism f^n and the minimality of $\sum_{x,y\in S_0} d_G(x,y)$ implies that S_n and S_0 induce isomorphic subgraphs. Consequently, for every nonnegative integers n and m with $n < m$, the restriction of f^{m-n} to S_n is an isomorphism of $G[S_n]$ onto $G[S_m]$.

1. Suppose that there exists some integer l such that $S_{k+1} \subseteq B_l$ for every $k \geqslant l$.
   In this case, there surely exist two integers n and m ($n < m$) such that $S_m \subseteq S_n$ since B_l is finite. Thus $\bigcup_{n\leqslant j\leqslant m} S_j$ is a finite nonempty subset of $V(G)$ which is strictly stabilized by f and the result of the theorem is established.
2. Otherwise, for every $l \geqslant 0$, there exists $k \geqslant l$ such that $S_{k+1} \not\subseteq B_l$.
   Let k be the least integer satisfying such a condition for $l = 0$; we will prove that $k = 0$. Let $x \in S_k$. Since $d_G(f(x), B_0) \leqslant 1$ for every $x \in S_k$, $\varphi_0 \circ f(S_k) \subseteq S_k$. The strict inclusion $\varphi_0 \circ f(S_k) \subset S_k$ implies that a proper subset of S_k is quasi-invariant and this facts also contradicts the minimality of $|S_0|$. Hence $\varphi_0 \circ f(S_k) = S_k$ and the restrictions of φ_0 and f to $f(S_k)$ and S_k, respectively, are isomorphisms; since φ_0 and proj_{B_0} coincide on $f(S_k)$, it results that $S_{k+1} \cap B_0 = \emptyset$, and B_{k+1} and B_0 are strictly parallel prefibers inducing boxes of G.
   Suppose that $k > 0$ (which implies that $S_1 \subseteq B_0$). Since $S_k \cap S_{k+1} = \emptyset$ and each element of S_{k+1} is at distance 1 from some element of S_k, then $S_1 \cap S_0 = \emptyset$ and each element of S_1 is at distance 1 from some element of S_0.
   Besides, since S_1 is contained in $B_0 = \mathrm{pref}(S_0)$, by Chastand [10, Lemma 7.7], there exist an edge $\{a,b\} \subseteq S_0$ and an edge $\{c,d\}$ with $c \in S_0$ and $d \in S_1$ which have the same colour (see [10, Section 6] for the description of the edge colouring of a fiber-complemented graph associated to the Djokovic relation of its edge set). Put $a' := f^k(a)$, $b' := f^k(b)$, $c' := f^k(c)$ and $d' := f^k(d)$; thus, $\{a',b'\}$ is an edge of $G[S_k]$ and $\{c',d'\}$ is an edge of G with $c' \in S_k$ and $d' \in S_{k+1}$ since the restriction of f^k to $S_0 \cup S_1$ is an isomorphism. Moreover, note that $\{a,b\}$ and $\{a',b'\}$ have the same colour since they belong to the same class of the Djokovic relation, and $\{c,d\}$ and $\{c',d'\}$ too, which contradicts the fact that $\{a',b'\}$ and $\{c',d'\}$ do not have the same colour by Chastand [10, Lemma 7.7] because $d' \notin B_0$.

Consequently, no edge $\{c,d\}$ with $c\in S_0$ and $d\in S_1$ is contained in $B_0=\mathrm{pref}(S_0)$, which contradicts the hypothesis $S_1\subseteq B_0$. Thus $k=0$. Moreover, for every $n\geqslant 0$, $S_{n+1}\cap B_n=\emptyset$, and B_n and B_{n+1} are parallel prefibers with $S_n=\mathrm{proj}_{B_n}(S_{n+1})$ and $S_{n+1}=\mathrm{proj}_{B_{n+1}}(S_n)$.

3. Let $u_0\in S_0$, and let $(u_n)_{n\geqslant 0}$ be the sequence of vertices defined by $u_{n+1}:=\mathrm{proj}_{B_{n+1}}(u_n)$. For every integers i and j, $u_j=\mathrm{proj}_{B_j}(u_i)$, $f(u_j)=\mathrm{proj}_{B_{j+1}}(f(u_i))$ and $d_G(u_i,u_j)=d_G(B_i,B_j)$ since B_i and B_j are parallel prefibers.
The absence of isometric ray in G implies that the sequence $(u_n)_{n\geqslant 0}$ induces no isometric ray and there exist some integers k and n with

$$d_G(u_n,u_{n+k})\geqslant d_G(u_n,u_{n+k+1}). \tag{I}$$

Let k be the least integer such that there exists n satisfying (I).
If $k=0$, then $S_n=S_{n+1}$. Thus S_n is strictly stabilized by f and the theorem is established. Otherwise, with this integer k, let n be the least nonnegative integer satisfying (I). Then every path $\langle u_{n+j},u_{n+j+1},\ldots,u_{n+l}\rangle$ with length at most k is isometric.
Consequently, for every $j\leqslant 0$,

(a) $d_G(u_{n+j},u_{n+j+k})=d_G(u_{n+j+1},u_{n+j+k+1})$ and $d_G(u_{n+j+1},u_{n+j+k})=k-1$;
(b) $d_G(u_{n+j},u_{n+j+k+1})=k$ or $k-1$, since $d_G(u_{n+j},u_{n+j+k})\geqslant d_G(u_{n+j},u_{n+j+k+1})$;
(c) the edges $\{u_{n+j},u_{n+j+1}\}$ and $\{u_{n+j+k},u_{n+j+k+1}\}$ belong to the same color class since they are linked by the Djokovic relation (see [10, Section 6]).

Therefore,

(a) if $j=0$, then the edges $\{u_n,u_{n+1}\}$ and $\{u_{n+k},u_{n+k+1}\}$ are contained in parallel elementary prefibers, and u_{n+k+1} belongs to the prefiber generated by the set $\{u_{n+l};\ 0\leqslant l\leqslant k\}$, which is also equal to $\mathrm{pref}(\{u_n,u_{n+k}\})$ since the path $\langle u_n,u_{n+1},\ldots,u_{n+k}\rangle$ is isometric;
(b) similarly, for every $j\geqslant 0$, $u_{n+l+k+1}$ belongs to $\mathrm{pref}(\{u_{n+j+l};\ 0\leqslant l\leqslant k\})$, and by immediate induction, it belongs to $\mathrm{pref}(\{u_n,u_{n+k}\})$.

Since S_0 is finite, we deduce that there exist two integers n and k with $0\leqslant n\leqslant k$, independent from u_0, such that, for every $p\geqslant n$, $S_p\subseteq\mathrm{pref}(B_n\cup B_{n+k})$; this prefiber is finite by Corollary 4.1.1 and it contains a finite subset which is strictly stabilized by f, which completes the proof. □

Tardif [24] gave a simple example of an endomorphism which stabilizes no finite subset of any median graph with an isometric ray. This example works with any fiber-complemented graph with an isometric ray as well:

Let G be a fiber-complemented graph containing some isometric ray $P=\langle u_0,u_1,\ldots\rangle$; the endomorphism f of G defined for every $u\in V(G)$ by $f(u)=u_n$ with $n=d(u,u_0)+1$ stabilizes no subgraph of finite diameter.

Hence, from this fact and by Theorem 4.2.1, we deduce the following:

4.2.2. Theorem. *A moorable pre-median graph G without infinite elementary prefibers has the fixed box property* (*that is*, *every endomorphism of G strictly stabilizes a finite nonempty pre-median box*) *if and only if it contains no isometric rays* (*that is*, $(V(G),\mathscr{T})$ *is a compact Hausdorff space in the sense of* [10, *Theorem* 7.10]).

4.2.3. Corollary.

(a) *Every endomorphism of a median graph G strictly stabilizes a nonempty finite cube if and only if G contains no isometric rays* [24, *Theorem* 1.2(2)]
(b) *Every endomorphism of a quasi-median graph G without infinite simplices strictly stabilizes a nonempty finite Hamming graph if and only if G contains no isometric rays.*
(c) *Every endomorphism of a pseudo-median graph G* (*respectively*, *of a weakly median graph G*) *without infinite elementary prefibers strictly stabilizes a nonempty finite regular pseudo-median graph* (*respectively*, *a nonempty finite regular weakly median graph*) *if and only if G contains no isometric rays.*

Proof. The statements (a) and (b) are immediate consequences of Theorem 4.2.2 and of [10, Corollary 7.11].

The statement (c) on pseudo-median graphs results from [4, Theorem 12] (*Every endomorphism of a finite pseudo-median graph strictly stabilizes a regular nonempty pseudo-median subgraph*). According to Bandelt and Mulder [4], this subgraph is the Cartesian product of an hypercube and a regular elementary pseudo-median graph (a simplex or a simplex minus a perfect matching).

The statement on weakly median graphs results from a similar theorem in [2]. □

5. Commuting families of endomorphisms

5.1. Closed pre-median subgraphs

Let A be a subset of the vertex set of a fiber-complemented graph G. We denote by $\bar{A}$ the closure of A with respect to the topology $\mathscr{T}$ on $V(G)$. A subgraph of G is said to be a closed *subgraph* whenever its vertex set is closed with respect to $\mathscr{T}$.

5.1.1. Lemma. *If u and v are two vertices of an interval-finite fiber-complemented graph G, then the family $\mathscr{D}(u,v)$ of all copoints of G which contain u but not v is finite.*

Proof. Let $\{x,y\}$ be an edge of G and let S be the prefiber generated by this edge. Put $u' := \mathrm{proj}_S(u)$ and $v' := \mathrm{proj}_S(v)$. If $u' = v'$, then no copoint relative to S belongs to $\mathscr{D}(u,v)$. If $u' \neq v'$, then $\mathrm{proj}_S^{-1}(u')$ belongs to $\mathscr{D}(u,v)$ and S is surely parallel to some elementary prefiber S' which contains at least one edge of $G[I(u,v)]$. Moreover note that one can associate the same family of copoints with any other elementary prefiber S' parallel to S. Therefore the map $\{x,y\} \mapsto \mathrm{proj}_S^{-1}(u')$, where $S := \mathrm{pref}(\{x,y\})$ and $u' := \mathrm{proj}_S(u)$, from the edge set of $G[I(u,v)]$ onto $\mathscr{D}(u,v)$ is surjective. Since, by hypothesis, $I(u,v)$ is finite, $\mathscr{D}(u,v)$ is also finite. □

5.1.2. Lemma. *Let G be an interval-finite fiber-complemented graph without infinite elementary prefibers and let $A \subseteq V(G)$. A vertex u of G belongs to $\bar{A} - A$ if there exists an infinite sequence $(u_n)_{n \geqslant 0}$ of distinct elements of A such that $u \in I(u_i, u_j)$ for every (i,j) with $i \neq j$.*

Proof. Let $u \in \bar{A} - A$; we will construct a sequence $(u_n)_{n \geqslant 0}$ satisfying the conditions of the lemma. Choose an arbitrary u_0 in A. Let $m \geqslant 0$ and suppose that u_i is already defined for $0 \leqslant i \leqslant m$. Put $C := \bigcap \mathscr{D}_m(u)$ with $\mathscr{D}_m(u) := \bigcup_{0 \leqslant i \leqslant m} \mathscr{D}(u, u_i)$ (recall that $\mathscr{D}(u, u_i)$ is the family of copoints containing u but not u_i). With every copoint W of G, one can associate some elementary prefiber S and some family $\mathscr{F}$ of copoints such that $W \in \mathscr{F}$, for every $W' \in \mathscr{F}$, $|W' \cap S| = 1$ and $\mathscr{F}$ is a partition of $V(G)$ (see [10, Section 7]). Since G contains no infinite elementary prefibers, this family $\mathscr{F}$ is finite; thus $V(G) - W$ is a finite union of copoints and the family $\mathscr{D}_m(u)$ is finite as well by Lemma 5.1.1. Hence, $V(G) - C = \bigcup_{W \in \mathscr{D}_m(u)} (V(G) - W)$ is a finite union of copoints and it is a closed set. Consequently, C is an open set containing u. Besides, $C \cap A \neq \emptyset$ since $u \in \bar{A}$. Choose u_{m+1} in $C \cap A$; obviously, this vertex is different from u_i for $0 \leqslant i \leqslant m$.

We will prove that u belongs to $I(u_i, u_{m+1})$, with $0 \leqslant i \leqslant m$. For such an integer i, there exists a copoint W in $\mathscr{D}(u, u_i)$ with $\mathrm{proj}_W(u_i) = u$ (for instance, choose a copoint associated with the elementary prefiber generated by $\{u, w_i\}$, where w_i is a neighbour of u in $I(u, u_i)$); it results that for every i, $0 \leqslant i \leqslant m$, $\mathrm{proj}_C(u_i) = u$ and $u \in I(u_i, u_{m+1})$, by the definition of a prefiber in [10, Section 3]. $\square$

5.1.3. Theorem. *Every pre-median subgraph of an interval-finite pre-median graph without infinite elementary prefibers is a closed subgraph.*

Proof. Let H be a pre-median subgraph of any interval-finite pre-median graph G and suppose that $V(H)$ is not a closed set. We will prove that G contains an infinite elementary prefiber.

By Lemma 5.1.2, there exist a vertex u of $G - H$ and an infinite sequence $(u_n)_{n \geqslant 0}$ of vertices of H such that $u \in I(u_i, u_j)$ for every (i,j) with $i \neq j$. This vertex u does not belong to $V(H)$ and H is a pre-median subgraph of G; thus by Definition 3.4.2, u is not the unique median of some triple (u_i, u_j, u_k) of elements of the sequence. For any fixed integers i and j and for every integer k, the triple (u_i, u_j, u_k) has at least two medians u and v_k; since the interval $I(u_i, u_j)$ is finite by hypothesis, there exist an infinite set of non-negative integers K and a vertex v of $I(u_i, u_j) - \{u\}$ such that, for every $k \in K$, the vertices u and v are distinct medians of (u_i, u_j, u_k). For every $k \in K$, let (x_k, y_k, u'_k) be a quasi-median (possibly a median) of the triple (u, v, u_k). Since the interval $I(u, v)$ is finite, there exist an infinite set of non-negative integers $L \subseteq K$ and two vertices u' and v' of some uv-geodesic with, for every $k \in L$, $u' = x_k$ and $v' = y_k$. Note that, even if k_1 and k_2 are distinct elements of L, then $u = u'$, $v = v'$ and $u'_k \neq u'_{k_2}$ since u and v belong to some $u_{k_1}u_{k_2}$-geodesic. Consequently, there exists an infinite family $(u'_k)_{k \in L}$ of pairwise distinct vertices such that (u, v, u'_k) is the quasi-median of (u, v, u_k), with the same size $l = d(u, v)$.

If $l=1$, then, for every $k\in L$, u'_k belongs to the Δ-closure of the edge $\{u,v\}$; thus the elementary prefiber $\mathrm{pref}(\{u,v\})$ is infinite (recall that every prefiber contains the Δ-closure of any of its edges).

Otherwise, by Chepoi [13, Theorem 2], $d(u'_k,x)=l$ for every $k\in L$ and for every $x\in I(u,v)$. Since the interval $I(u,v)$ is finite, for every $k\in L$, there exists an edge $\{x,y\}$ in some uv-geodesic with $d(u'_k,x)=d(u'_k,y)=l\geqslant 2$, and by the ∇-property, there exists a vertex u''_k adjacent to x and y with $d(u'_k,u''_k)=l-1$. Note that necessarily the elements of the family $(u''_k)_{k\in L}$ are pairwise distinct; thus the Δ-closure of the edge $\{x,y\}$ is an infinite elementary prefiber since it contains the infinite family $(u''_k)_{k\in L}$.

It results that if G contains no infinite elementary prefiber, the pre-median subgraph H is closed, which proves the theorem. $\square$

5.2. Finite stabilized subgraphs

5.2.1. Theorem. *If G is a moorable compact Hausdorff pre-median graph, then the elements of every commuting family Φ of endomorphism strictly stabilize a common nonempty finite pre-median box.*

Proof. For any commuting family Ψ of endomorphisms of G, we note $G_\Psi := \bigcap_{f\in\Psi} G_f$ where G_f is the subgraph induced by the set of vertices of finite period under f (see Notation 4.1.2).

Let Ψ be a finite subfamily of Φ. We will show by induction on $|\Psi|$ that G_Ψ is a nonempty pre-median subgraph of G. If $|\Psi|=1$, then the result is a consequence of Theorem 4.2.2. Suppose that the result is true for some finite family Ψ and let $g\in\Phi-\Psi$. For every $f\in\Psi$ and for every $x\in G_\Psi$, $f^n(g(x))=g(f^n(x))=g(x)$; thus $g(x)\in G_\Psi$ and $g(G_\Psi)\subseteq G_\Psi$. It results that $(G_\Psi)_g=G_{\Psi\cup\{g\}}$ is a nonempty pre-median subgraph of G by Lemma 4.1.3.

Consequently, for every finite family $\Psi\subseteq\Phi$, G_Ψ is a nonempty pre-median subgraph of G. Furthermore, by hypothesis, G is a compact Hausdorff moorable pre-median graph; thus its elementary prefibers are finite by Chastand [10, Theorem 7.10], its intervals are finite by Proposition 4.1.1 and G_Ψ is a closed subgraph of G by Theorem 5.1.3. It follows that by compacticity $G_\Phi := \bigcap_{f\in\Phi} G_f$ is nonempty. By an argument similar to the proof of Lemma 4.1.3, one can establish that G_Φ is a pre-median subgraph of G and that the restriction of every element of Φ to G_Φ is an automorphism of G_Φ. Finally, by Chastand [10, Theorem 8.8] G_Φ contains a finite nonempty pre-median box which is strictly stabilized by every element of Φ. $\square$

With similar arguments as in Corollary 4.2.3, we complete this theorem by a corollary relative to regular strictly stabilized subgraphs in some subclasses.

5.2.2. Corollary.

(a) [24, Theorem 1.2(3)] *If Φ is a commuting family of endomorphisms of a median graph G without isometric rays, then the elements of Φ strictly stabilize a common nonempty finite cube.*

(b) *If Φ is a commuting family of endomorphisms of a quasi-median graph G without isometric rays and infinite simplices, then the elements of Φ strictly stabilize a common nonempty finite Hamming graph.*

(c) *If Φ is a commuting family of endomorphisms of a pseudo-median graph G (resp., a weakly median graph) without isometric rays and infinite elementary prefibers, then the elements of Φ strictly stabilize a common nonempty finite regular pseudo-median graph (resp., a common nonempty finite regular weakly median graph).*

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